

On a fractional class of analytic function defined by using a new operator

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Abstract

In this article, we impose a new class of fractional analytic functions in the open unit disk. By considering this class, we define a fractional operator, which is generalized Salagean and Ruscheweyh differential operators. Moreover, by means of this operator, we introduce an interesting subclass of functions which are analytic and univalent. Furthermore, this effort covers coefficient bounds, distortions theorem, radii of starlikeness, convexity, bounded turning, extreme points and integral means inequalities of functions belongs to this class. Finally, applications involving certain fractional operators are illustrated.

Keywords: Fractional analytic functions; univalent function; fractional calculus; unit disk; subordination and superordination

1 Introduction and preliminaries

Recently, One of the substantive issues in many applications of geometric function theory is how to employ the fractional operators to analytic and univalent functions and what the advantages for this utilized. In other hand, make use of functional analytic functions to define fractional operators and what of the results of this. So far, many mathematicians in different stages considered this issues and gave numerous applications based on certain fractional operators of analytic function in physics, engineering and mathematical applications (see [1]).

In the theory of geometric functions, the Koebe function is formulated by

$$f(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} n z^n.$$

The rotated Koebe function is

$$f_{\alpha}(z) = \frac{z}{(1-\alpha z)^2} = \sum_{n=1}^{\infty} n \alpha^{n-1} z^n$$

with a complex number of absolute value 1. The Koebe function and its rotations are schlicht: that is, univalent (analytic and one-to-one) and achieving $f(0) = 0$ and $f'(0) = 1$. Srivastava et al [2], introduced a fractional analytic function as follows:

$$f(z) = \frac{z^{\alpha+1}}{(1-z)^\alpha}, \quad \alpha \in \mathbb{R}.$$

In this effort, we define a class \mathcal{A}_μ of functional fractional analytic functions $F_\mu(z)$ in unit disk $\mathbb{U} := \{z \in \mathbb{C}; |z| < 1\}$ as follows:

$$F_\mu(z) = \frac{z^\mu}{1-z^\mu}, \quad (1.1)$$

where $\mu := \frac{n+m-1}{m}$, $n, m \in \mathbb{N}$. Hence, $\mu = 1$, when $n = 1$ and has the power series formal:

$$F_\mu(z) = z + \sum_{n=2}^{\infty} a_n z^{\mu n} \quad (1.2)$$

$$(\mu \geq 1; n \in \mathbb{N}, z \in \mathbb{U}),$$

which normalize by $F_\mu(z)|_{z=0} = 1$ and $F'_\mu(z)|_{z=0} = 1$ for all $z \in \mathbb{U}$.

Recall that a function $F_\mu \in \mathcal{A}_\mu$ is called bounded turning if it satisfies the following inequality:

$$\Re\{F'_\mu(z)\} > \psi \quad (0 \leq \psi < 1), \quad (1.3)$$

and a function $F_\mu \in \mathcal{A}_\mu$ is starlike function in \mathbb{U} if satisfies

$$\Re\left\{\frac{zF'_\mu(z)}{F_\mu(z)}\right\} > \psi \quad (0 \leq \psi < 1). \quad (1.4)$$

Furthermore, a function $F_\mu \in \mathcal{A}_\mu$ is convex function in \mathbb{U} if satisfies

$$\Re\left\{1 + \frac{zF''_\mu(z)}{F'_\mu(z)}\right\} > \psi \quad (0 \leq \psi < 1). \quad (1.5)$$

(see, for more details [3] and [4]).

Next, if the function $F_\mu(z)$ of form (1.2) and $G_\mu(z) = z + \sum_{n=2}^{\infty} b_n z^{\mu n}$ are two functions in class \mathcal{A}_μ , then the convolution (or Hadamard product) of two analytic functions is denoted by $F_\mu * G_\mu$ and is given by

$$F_\mu(z) * G_\mu(z) = z + \sum_{n=2}^{\infty} a_n b_n z^{\mu n}$$

and satisfy

$$[F_\mu(z) * G_\mu(z)]|_{z=0} = 0 \quad \text{and} \quad [(F_\mu(z) * G_\mu(z))']|_{z=0} = 1.$$

Now let a functional function $\Theta_\mu(z)$ defined as follows:

$$\begin{aligned} \Theta_\mu(z) &= \frac{\mu z^\mu}{(1-z^\mu)^2} + \frac{\mu z^\mu}{(1-z^\mu)}, \\ &= z + \sum_{n=2}^{\infty} (\mu n) z^{\mu n}, \quad (z \in \mathbb{U}). \end{aligned} \quad (1.6)$$

By employing method of the convolution product of analytic function $\Theta_\mu(z)$, we obtain

$$\begin{aligned}\Theta_{\mu,k}(z) &= \underbrace{F_\mu(z) * \cdots * F_\mu(z)}_{k\text{-times}}, \\ &= z + \sum_{n=2}^{\infty} (\mu n)^k z^{\mu n}, \quad (z \in \mathbb{U}).\end{aligned}\tag{1.7}$$

For $F_\mu(z) \in \mathcal{A}_\mu$, we define differential operator $D_\mu^k F_\mu(z)$ as the following

$$D_\mu^k F_\mu(z) = \Theta_{\mu,k}(z) * F_\mu(z), \quad (|z| < 1).$$

where

$$\begin{aligned}D_\mu^0 F_\mu(z) &= F_\mu(z), \\ D_\mu^1 F_\mu(z) &= z F'_\mu(z) = z + \sum_{n=2}^{\infty} (\mu n) a_n z^{\mu n} \\ D_\mu^2 F_\mu(z) &= D(D_\mu F_\mu(z)) = z + \sum_{n=2}^{\infty} (\mu n)^2 a_n z^{\mu n} \\ &\vdots \\ D_\mu^k F_\mu(z) &= D(D_\mu^{k-1} F_\mu(z)) = z + \sum_{n=2}^{\infty} (\mu n)^k a_n z^{\mu n}.\end{aligned}$$

In general, we write

$$D_\mu^\beta F_\mu(z) = z + \sum_{n=2}^{\infty} (\mu n)^\beta a_n z^{\mu n},\tag{1.8}$$

$$(\mu \geq 1; n \in \mathbb{N} \setminus \{1\}; \beta \in \mathbb{N}_0; z \in \mathbb{U}).$$

Based of above functional function, we deal with a new operator $D_\mu^\beta F_\mu(z)$ which is a generalization of the well known operators such as Ruscheweyh differential operators in \mathbb{U} (see [5]).

With recalling the principle of subordination between two analytic functions f and g in open unit disk \mathbb{U} (see [6] and [7]), the function f is *subordinate* to g if there exists a Schwarz function $w(z)$, analytic in \mathbb{U} with $w(z)|_{z=0} = 0$ and $|w(z)| = 1$ for all $z \in \mathbb{U}$ such that

$$f(z) = g(w(z)), \quad (z \in \mathbb{U}).$$

In addition, we write this subordination by

$$f(z) \prec g(z) \quad (z \in \mathbb{U}).\tag{1.9}$$

In special case, if the function g is univalent in \mathbb{U} then the above subordination is equivalent to

$$f(z)|_{z=0} = g(z)|_{z=0} \quad \text{and} \quad f(\mathbb{U}) \subset g(\mathbb{U}).$$

Now we define the following a new class of analytic functions and investigate several interesting results.

Definition 1. Let the functions

$$\Phi_\mu(z) = z + \sum_{n=2}^{\infty} \vartheta_n z^{\mu n} \quad \text{and} \quad \Psi_\mu(z) = z + \sum_{n=2}^{\infty} \lambda_n z^{\mu n}$$

be analytic in the open unit disk \mathbb{U} where

$$\vartheta_n \geq 0, \quad \lambda_n \geq 0 \quad \text{and} \quad \vartheta_n \geq \lambda_n \quad (n \in \mathbb{N} \setminus \{0, 1\}).$$

Then a function $F_\mu(z) \in \mathcal{A}_\mu$ is said to be in the class $\mathcal{E}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$ if and only if

$$\frac{D_\mu^k(F_\mu * \Phi_\mu)(z)}{D_\mu^m(F_\mu * \Psi_\mu)(z)} \prec (1 - \gamma) \frac{1 + Az}{1 + Bz} + \gamma \quad (z \in \mathbb{U}),$$

where \prec represents the subordination in (1.9), $F_\mu(z) * \Psi_\mu(z) \neq 0$, A and B are arbitrarily fixed numbers such that

$$-1 \leq B < A \leq 1 \quad \text{and} \quad -1 \leq B < 0$$

with

$$0 \leq \gamma < 1 \quad \text{and} \quad k \geq m \quad (k, m \in \mathbb{N}_0).$$

In other words, $F_\mu \in \mathcal{E}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$ if and only if there exists an analytic function $w(z)$ satisfying

$$w(z)|_{z=0} = 0 \quad \text{and} \quad |w(z)| = 1 \quad (z \in \mathbb{U})$$

such that

$$\frac{D_\mu^k(F_\mu * \Phi_\mu)(z)}{D_\mu^m(F_\mu * \Psi_\mu)(z)} = (1 - \gamma) \frac{1 + A\omega(z)}{1 + B\omega(z)} + \gamma \quad (z \in \mathbb{U}). \quad (1.10)$$

The condition (1.10) can be expressed by the equivalent inequality

$$\left| \frac{\frac{D_\mu^k(F_\mu * \Phi_\mu)(z)}{D_\mu^m(F_\mu * \Psi_\mu)(z)} - 1}{(A - B)(1 - \gamma) - B \left(\frac{D_\mu^k(F_\mu * \Phi_\mu)(z)}{D_\mu^m(F_\mu * \Psi_\mu)(z)} - 1 \right)} \right| < 1 \quad (z \in \mathbb{U}). \quad (1.11)$$

Let \mathcal{X}_μ be the class of analytic functions $F_\mu(z)$ in unit disk \mathbb{U} of the following form

$$F_\mu(z) = z - \sum_{n=2}^{\infty} a_n z^{\mu n}, \quad (1.12)$$

$$(a_n \geq 0; \mu \geq 1; n \in \mathbb{N} \setminus \{1\})$$

which satisfies $F_\mu(z)|_{z=0} = 0$ and $F'_\mu(z)|_{z=0} = 1$ for all $z \in \mathbb{U}$. We denote by $\tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \gamma, \mu)$ the subclass of functions in $\mathcal{E}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \gamma, \mu)$ that has their non-zero coefficients, from second onwards, all negative. Further let

$$\tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma) = \mathcal{E}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma) \cap \mathcal{X}_\mu.$$

For suitable choices of Φ and Ψ , we definitely obtain the function subclasses of \mathcal{A}_μ . For example, we have the following:

$$\tilde{\mathcal{E}}_{0,0}\left(\frac{\mu z^\mu}{(1 - z^\mu)^2}, \frac{z^\mu}{1 - z^\mu}, 1, -1, \mu, \gamma\right) = \mathcal{S}_\mu^*(\gamma) \quad (1.13)$$

and

$$\tilde{\mathcal{E}}_{0,0}(\frac{\mu^2 z^\mu + \mu^2 z^{2\mu}}{(1-z^\mu)^3}, \frac{\mu z^\mu}{(1-z^\mu)^2}, 1, -1, \mu, \gamma) = \mathcal{K}_\mu(\gamma) \quad (1.14)$$

If $\mu = 1$ in (1.13) and (1.14), respectively we obtain the well known subclasses:

$$\mathcal{S}^*(\gamma) \quad \text{and} \quad \mathcal{K}(\gamma),$$

which were investigated by [8].

2 Characterization properties

In this section, we consider several properties for the function $F_\mu(z) \in \tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$. We will divide this section into five subsections.

2.1 Coefficient Bounds

Theorem 1. If $F_\mu(z) \in \mathcal{A}_\mu$ satisfies the following inequality

$$\sum_{n=2}^{\infty} [(1-B) \left((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n \right) + (A-B)(1-\gamma)(\mu n)^m \lambda_n] |a_n| \leq (A-B)(1-\gamma). \quad (2.1)$$

$$(\vartheta_n \geq 0; \lambda_n \geq 0; \vartheta_n \geq \lambda_n \ (n \in \mathbb{N} \setminus \{1\}); \mu \geq 1; 0 \leq \gamma < 1; k \geq m; k, m \in \mathbb{N}_0).$$

Then

$$F_\mu(z) \in \mathcal{E}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma).$$

Proof. Let the condition (2.1) holds, then we obtain

$$\begin{aligned} & \left| D_\mu^k(F_\mu * \Phi_\mu)(z) - D_\mu^m(F_\mu * \Psi_\mu)(z) \right| \\ & \quad - \left| (A-B)(1-\gamma) D_\mu^m(F_\mu * \Psi_\mu)(z) - B \left(D_\mu^k(F_\mu * \Phi_\mu)(z) - D_\mu^m(F_\mu * \Psi_\mu)(z) \right) \right| \\ & = \left| \sum_{n=2}^{\infty} \left((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n \right) a_n z^{\mu n} \right| - \left| (A-B)(1-\gamma) z + (A-B) \right. \\ & \quad \times (1-\gamma) \sum_{n=2}^{\infty} (\mu n)^m \vartheta_n a_n z^{\mu n} - B \sum_{n=2}^{\infty} \left((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n \right) a_n z^{\mu n} \left. \right| \\ & \leq \sum_{n=2}^{\infty} \left((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n \right) |a_n| r^{\mu n} + (A-B)(1-\gamma) r \\ & \quad + (A-B)(1-\gamma) \sum_{n=2}^{\infty} (\mu n)^m \vartheta_n |a_n| r^{\mu n} + |B| \sum_{n=2}^{\infty} \left((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n \right) |a_n| r^{\mu n} \quad (2.2) \\ & \leq \sum_{n=2}^{\infty} [(1-B) \left((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n \right) \\ & \quad + (A-B)(1-\gamma)(\mu n)^m \vartheta_n] |a_n| - (A-B)(1-\gamma) \leq 0. \quad (2.3) \end{aligned}$$

For all $r(0 \leq r < 1)$ the inequality in (2.2) holds true. Thus, letting $r \rightarrow 1-$ in (2.2), we obtain (2.3) Hence, this completes proof of Theorem 1. \square

Theorem 2. If $F_\mu(z) \in \mathcal{X}_\mu$ satisfies the following inequality

$$\sum_{n=2}^{\infty} [(1-B) ((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n] a_n \leq (A-B)(1-\gamma). \quad (2.4)$$

$$(\vartheta_n \geq 0; \lambda_n \geq 0; \vartheta_n \geq \lambda_n \ (n \in \mathbb{N} \setminus \{1\}); \mu \geq 1; 0 \leq \gamma < 1; k \geq m; k, m \in \mathbb{N}_0).$$

Then

$$F_\mu(z) \in \tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma).$$

Proof. Since

$$\tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma) \subset \mathcal{E}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma),$$

we only need to prove the *only if* part of Theorem 2 for function $F_\mu(z) \in \mathcal{X}_\mu$ we can write

$$\begin{aligned} & \left| \frac{(D_\mu^k(F_\mu * \Phi_\mu)(z))/(D_\mu^m(F_\mu * \Psi_\mu)(z)) - 1}{(A-B)(1-\gamma) - B((D_\mu^k(F_\mu * \Phi_\mu)(z))/(D_\mu^m(F_\mu * \Psi_\mu)(z)) - 1)} \right| \\ &= \left| \frac{D_\mu^k(F_\mu * \Phi_\mu)(z) - D_\mu^m(F_\mu * \Psi_\mu)(z)}{(A-B)(1-\gamma)D_\mu^m(F_\mu * \Psi_\mu)(z) - B[D_\mu^k(F_\mu * \Phi_\mu)(z) - D_\mu^m(F_\mu * \Psi_\mu)(z)]} \right| \\ &= \left| \frac{\sum_{n=2}^{\infty} ((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) a_n z^{\mu n-1}}{(A-B)(1-\gamma) - (A-B)(1-\gamma) \sum_{n=2}^{\infty} (\mu n)^m \lambda_n a_n z^{\mu n-1} + B \sum_{n=2}^{\infty} ((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) a_n z^{\mu n-1}} \right| < 1. \end{aligned}$$

Since $\Re(z) \leq |z|$ for all $z \in \mathbb{U}$, we have

$$\Re \left\{ \frac{\sum_{n=2}^{\infty} ((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) a_n z^{\mu n-1}}{(A-B)(1-\gamma) - (A-B)(1-\gamma) \sum_{n=2}^{\infty} (\mu n)^m \lambda_n a_n z^{\mu n-1} + B \sum_{n=2}^{\infty} ((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) a_n z^{\mu n-1}} \right\} < 1.$$

If we choose z to be real and let $z \rightarrow 1-$, we obtain

$$\sum_{n=2}^{\infty} [(1-B) ((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n] a_n \leq (A-B)(1-\gamma).$$

which is equivalent to (2.4). The result is sharp for functions F given by

$$F_\mu(z) = z - \frac{(A-B)(1-\gamma)}{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]} z^{\mu n} \ (n \geq 2). \quad (2.5)$$

This completes the proof of Theorem 2. \square

Corollary 1. Let a function $F_\mu(z)$ defined by (1.12) belongs to $\tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$. Then

$$a_n \leq \frac{(A-B)(1-\gamma)}{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]} \quad (n \geq 2). \quad (2.6)$$

Remark 1. By taking different choices for the functions $\Phi_\mu(z)$ and $\Psi_\mu(z)$ same as stated in (1.13) and (1.14), Theorem 2 leads us to the necessary and sufficient conditions for a function $F_\mu(z)$ to be in the following classes:

$$\mathcal{S}_\mu^*(\gamma) \quad \text{and} \quad \mathcal{K}_\mu(\gamma).$$

2.2 Distortion theorems

Theorem 3. Let the function $F_\mu(z)$ defined by (1.12) be in $\tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$. Then

$$\begin{aligned} |F_\mu(z)| &\geq |z| - \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} |z|^{2\mu} \\ &\leq |z| + \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} |z|^{2\mu}. \end{aligned} \quad (2.7)$$

The result is sharp.

Proof. By considering Theorem 1, since

$$\Xi(n) = [(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]$$

is an increasing function of n ($n \geq 2$), we get

$$\Xi(2) \sum_{n=2}^{\infty} |a_n| \leq \sum_{n=2}^{\infty} \Xi(n) |a_n| \leq (A-B)(1-\gamma),$$

that is

$$\sum_{n=2}^{\infty} |a_n| \leq \frac{(A-B)(1-\gamma)}{\Xi(2)}.$$

Therefore, we have

$$\begin{aligned} |F_\mu(z)| &\leq |z| + |z|^{2\mu} \sum_{n=2}^{\infty} |a_n|, \\ |F_\mu(z)| &\leq |z| + \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} |z|^{2\mu}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} |F_\mu(z)| &\geq |z| - \sum_{n=2}^{\infty} |a_n| |z|^{\mu n} \geq |z| - |z|^{2\mu} \sum_{n=2}^{\infty} |a_n| \\ &\geq |z| - \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} |z|^{2\mu}. \end{aligned}$$

The result is sharp for the function

$$F_\mu(z) = |z| - \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} |z|^{2\mu}. \quad (2.8)$$

This completes the proof of Theorem 3. \square

Theorem 4. Let $F_\mu(z)$ defined by (1.12) be in the class $\tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$, then

$$|F'_\mu(z)| \geq 1 - \frac{2\mu(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} |z|^{2\mu-1} \quad (2.9)$$

$$\leq 1 + \frac{2\mu(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} |z|^{2\mu-1} \quad (2.10)$$

This result is sharp.

Proof. Similarly $\Xi(n)/n$ is an increasing function of $n(n \geq 1)$,

$$\frac{\Xi(2)}{2\mu} \sum_{n=2}^{\infty} (\mu n) |a_n| \leq \sum_{n=2}^{\infty} \frac{\Xi(n)}{(\mu n)} (\mu n) |a_n| = \sum_{n=2}^{\infty} \Xi(n) |a_n| \leq (1-\gamma)(A-B), \quad (2.11)$$

that is

$$\sum_{n=2}^{\infty} (\mu n) |a_n| \leq \frac{2\mu(1-\gamma)(A-B)}{\Xi(2)}. \quad (2.12)$$

Then, we get

$$\begin{aligned} |F'_\mu(z)| &\leq 1 + |z|^{2\mu-1} \sum_{n=2}^{\infty} (\mu n) |a_n| \\ &\leq 1 + \frac{2\mu(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} |z|^{2\mu-1}. \end{aligned} \quad (2.13)$$

and similarly

$$\begin{aligned} |F'_\mu(z)| &\geq 1 - |z|^{2\mu-1} \sum_{n=2}^{\infty} (\mu n) |a_n| \\ &\geq 1 - \frac{2\mu(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} |z|^{2\mu-1}. \end{aligned} \quad (2.14)$$

It is clear that the assertions of Theorem 4 are sharp for the function $F_\mu(z)$ given by (2.8). This complete the proof of Theorem 4. \square

2.3 The radii subclasses of class $\tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$

In this section radii of bounded turning, convexity and starlikeness for functions $F_\mu(z) \in \tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$ are studied.

Theorem 5. Let $F_\mu(z)$ given by (1.12) be in the class $\tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$, then

i- $F_\mu(z)$ is starlike of order ψ ($0 \leq \psi < 1$) in $|z| < r_1$, where

$$\begin{aligned} r_1 = \inf_{n \geq 2} \left\{ \frac{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]}{(A-B)(1-\gamma)} \right. \\ \left. \times \left(\frac{1-\psi}{\mu n - \psi} \right) \right\}^{1/(\mu n-1)}. \end{aligned} \quad (2.15)$$

ii- $F_\mu(z)$ is convex of order ψ ($0 \leq \psi < 1$) in $|z| < r_2$, where

$$\begin{aligned} r_2 = \inf_{n \geq 2} \left\{ \frac{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]}{(A-B)(1-\gamma)} \right. \\ \left. \times \left(\frac{1-\psi}{\mu n(\mu n - \psi)} \right) \right\}^{1/(\mu n-1)}, \end{aligned} \quad (2.16)$$

iii- $F_\mu(z)$ is close to convex of order ψ ($0 \leq \psi < 1$) in $|z| < r_3$, where

$$\begin{aligned} r_3 = \inf_{n \geq 2} \left\{ \frac{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]}{(A-B)(1-\gamma)} \right. \\ \left. \times \left(\frac{1-\psi}{\mu n} \right) \right\}^{1/(\mu n-1)}, \end{aligned} \quad (2.17)$$

Each the results is sharp for the function $F_\mu(z)$ given by (2.5)

Proof. It is sufficient show that

$$\left| \frac{zF'_\mu(z)}{F_\mu(z)} \right| \leq 1 - \psi, \quad \text{for } |z| < r_1, \quad (2.18)$$

where r_1 is defined by (2.15). Further, we find from (1.12) that

$$\left| \frac{zF'_\mu(z)}{F_\mu(z)} - 1 \right| \leq \frac{\sum_{n=2}^{\infty} (\mu n - 1) a_n |z|^{\mu n - 1}}{1 - \sum_{n=2}^{\infty} a_n |z|^{\mu n - 1}}. \quad (2.19)$$

Therefore, we satisfy (2.18) if and only if

$$\sum_{n=2}^{\infty} \frac{(\mu - \psi) a_n |z|^{\mu n - 1}}{(1 - \psi)} \leq 1. \quad (2.20)$$

Nevertheless, from Theorem 1, inequality (2.20) it will be true that if

$$\frac{(\mu - \psi) |z|^{\mu n - 1}}{(1 - \psi)} \leq \frac{[(1 - B) ((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A - B)(1 - \gamma)(\mu n)^m \vartheta_n]}{(A - B)(1 - \gamma)}$$

this is, if

$$|z| \leq \left\{ \frac{[(1 - B) ((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A - B)(1 - \gamma)(\mu n)^m \vartheta_n]}{(A - B)(1 - \gamma)} \times \left(\frac{1 - \psi}{\mu n - \psi} \right) \right\}^{1/(\mu n - 1)} \quad (2.21)$$

Or equivalent to

$$r_1 = \inf_{n \geq 2} \left\{ \frac{[(1 - B) ((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A - B)(1 - \gamma)(\mu n)^m \vartheta_n]}{(A - B)(1 - \gamma)} \times \left(\frac{1 - \psi}{\mu n - \psi} \right) \right\}^{1/(\mu n - 1)}. \quad (2.22)$$

This completes the proof of (2.15). To prove (2.16) and (2.17); respectively it is sufficient to show that

$$\left| 1 + \frac{zF''_\mu(z)}{F'_\mu(z)} - 1 \right| \leq 1 - \psi \quad (|z| \leq r_2; \quad 0 \leq \psi < 1) \quad (2.23)$$

and

$$|F'_\mu(z) - 1| \leq 1 - \psi \quad (|z| \leq r_3; \quad 0 \leq \psi < 1). \quad (2.24)$$

□

2.4 Extreme point

Theorem 6. Let $F_1(z) = z$, and

$$F_{\mu n}(z) = z - \frac{(A-B)(1-\gamma)}{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]} z^{\mu n} \quad (2.25)$$

Then $F_{\mu}(z) \in \tilde{\mathcal{E}}_{k,m}(\Phi, \Psi, A, B, \mu, \gamma)$ if and only if it can be expressed in the following form:

$$F(z) = \sum_{n=1}^{\infty} \eta_n F_{\mu n}(z),$$

where

$$\eta_n \geq 0, \quad \text{and} \quad \sum_{n=1}^{\infty} \eta_n = 1.$$

Proof. Assuming that,

$$F_{\mu}(z) = \sum_{n=1}^{\infty} \eta_n F_{\mu n}(z) \quad (2.26)$$

$$= z - \sum_{n=1}^{\infty} \eta_n \frac{(A-B)(1-\gamma)}{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]} z^{\mu n} \quad (2.27)$$

Then, from Theorem 1, we obtain

$$\begin{aligned} & \sum_{n=2}^{\infty} \left([(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n] \right. \\ & \quad \times \left. \frac{(A-B)(1-\gamma)}{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]} \right) \\ & = (A-B)(1-\gamma) \sum_{n=2}^{\infty} \eta_n = (A-B)(1-\gamma)(1-\eta_1) \leq (A-B)(1-\gamma). \end{aligned}$$

Therefore, in view of Theorem 1, we find that $F(z) \in \tilde{\mathcal{E}}_{k,m}(\Phi, \Psi, A, B, \mu, \gamma)$. Conversely, let us suppose that $F(z) \in \tilde{\mathcal{E}}_{k,m}(\Phi, \Psi, A, B, \mu, \gamma)$, then, since

$$a_n \leq \frac{(A-B)(1-\gamma)}{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]} \quad (2.28)$$

by setting

$$\eta_n = \frac{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]}{(A-B)(1-\gamma)} a_n, \quad n = \{2, 3, \dots\}$$

and

$$\eta_1 = 1 - \sum_{n=2}^{\infty} \eta_n.$$

Therefore, we have

$$F(z) = \sum_{n=1}^{\infty} \eta_n F_{\mu n}(z).$$

By this, we complete the proof of Theorem 6. \square

Corollary 2. The extreme point of the class $\tilde{\mathcal{E}}_{k,m}(\Phi, \Psi, A, B, \mu, \gamma)$ are given by

$$F_{\mu n}(z) = z - \frac{(A-B)(1-\gamma)}{[(1-B)((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n]} z^{\mu n}.$$

2.5 Integral Means Inequality

In this section, we consider some result due to Littlewood subordination (see [9]).

Lemma 1. If the functions f and g are analytic in open unit disk \mathbb{U} with

$$f(z) \prec g(z) \quad (z \in \mathbb{U}), \quad (2.29)$$

then, for $q > 0$ and $z = e^{i\theta}$ ($0 < r < 1$),

$$\int_0^{2\pi} |f(z)|^q d\theta \leq \int_0^{2\pi} |g(z)|^q d\theta. \quad (2.30)$$

Now, let use Lemma1 to prove the following Theorem.

Theorem 7. Assume that $F_\mu(z) \in \tilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$, $q > 0$, $-1 \leq B < A \leq 1$, $k, m \in \mathbb{N}_0$, and $F_{2\mu}(z)$ is defined by

$$F_{2\mu}(z) = z - \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} z^{2\mu}. \quad (2.31)$$

Then $z = re^{i\theta}$ ($0 < r < 1$), we obtain

$$\int_0^{2\pi} |F_\mu(z)|^q d\theta \leq \int_0^{2\pi} |F_{2\mu}(z)|^q d\theta.$$

Proof. For $F_\mu(z) = z - \sum_{n=2}^{\infty} a_n z^{\mu n}$ ($a_n \geq 0$) and by (2.30) is equivalent to proof,

$$\begin{aligned} & \int_0^{2\pi} \left| 1 - \sum_{n=2}^{\infty} a_n z^{\mu n-1} \right|^q d\theta \\ & \leq \int_0^{2\pi} \left| 1 - \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} z^{2\mu-1} \right|^q d\theta \end{aligned} \quad (2.32)$$

By applying Lemma 1, it would suffice to show that

$$\begin{aligned} & 1 - \sum_{n=2}^{\infty} a_n z^{\mu n-1} \\ & \prec 1 - \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} z^{2\mu-1}. \end{aligned}$$

By putting

$$\begin{aligned} & 1 - \sum_{n=2}^{\infty} a_n z^{\mu n-1} \\ & = 1 - \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]} \Theta(z). \end{aligned}$$

and by using Theorem1, we have

$$\begin{aligned}
& \left| \sum_{n=2}^{\infty} \frac{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]}{(A-B)(1-\gamma)} a_n z^{2\mu-1} \right| \\
& \leq |z|^{2\mu-1} \sum_{n=2}^{\infty} \frac{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]}{(A-B)(1-\gamma)} a_n \\
& \leq |z|^{2\mu-1} \sum_{n=2}^{\infty} \frac{[(1-B)((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2]}{(A-B)(1-\gamma)} a_n \\
& \leq |z| \leq 1.
\end{aligned}$$

This complete the proof of this Theorem. \square

3 Distortion Applications

In this section, we prove some distortion theorems involving certain fractional calculus (Srivastava and Owa) operators for function $F(z)$ belonging to class $\tilde{\mathcal{E}}_{k,m}(\Phi, \Psi, A, B, \mu, \gamma)$. Now let recall the following definitions:

Definition 2. The fractional derivative of order δ is defined, for a function $f(z)$ by

$$\mathcal{D}_z^\delta f(z) = \frac{1}{\Gamma(1-\delta)} \frac{d}{dz} \int_0^z (z-\zeta)^{-\delta} f(\zeta) d\zeta \quad (0 \leq \delta < 1), \quad (3.1)$$

where the function $f(z)$ is constrained and the multiplicity of $(z-\zeta)^{-\delta}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$ (see [10], [11]).

Definition 3. The fractional integral of order δ is defined, for a function $f(z)$ by

$$\mathcal{I}_z^\delta f(z) = \frac{1}{\Gamma(\delta)} \frac{d}{dz} \int_0^z (z-\zeta)^{\delta-1} f(\zeta) d\zeta \quad (\delta > 0), \quad (3.2)$$

where the function $f(z)$ is analytic in a simply connected domain of the complex z -plane containing the origin, and the multiplicity of $(z-\zeta)^{-\delta}$ is removed by suggesting $\log(z-\zeta)$ to be real when $z-\zeta > 0$ (see [10], [11]).

Definition 4. (see [10]) Under the hypotheses of Definition 2, the fractional derivative of order δ is defined for a function $f(z)$, by

$$\mathcal{D}_z^{v+\delta} = \frac{d^v}{dz^v} \{\mathcal{D}_z^\delta\} \quad (0 \leq \delta < 1; v \in \{0, 1, 2, \dots\}). \quad (3.3)$$

Using Definitions 2, 3 and 4, we have

Lemma 2. For the function $z^{\mu n}$, ($\mu \geq 1; n \in \mathbb{N}_0$), $z \in \mathbb{U}$, we have

$$\mathcal{D}_z^\delta \{z^{\mu n}\} = \frac{\Gamma(\mu n + 1)}{\Gamma(\mu n + 1 - \delta)} z^{\mu n - \delta} \quad (0 \leq \delta < 1) \quad (3.4)$$

and

$$\mathcal{I}_z^\delta \{z^{\mu n}\} = \frac{\Gamma(\mu n + 1)}{\Gamma(\mu n + 1 + \delta)} z^{\mu n + \delta} \quad (0 \leq \delta < 1). \quad (3.5)$$

Theorem 8. Let the function $F_\mu(z)$ given by (1.12) be in the class $\widetilde{\mathcal{E}}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \mu, \gamma)$. Then

$$|\mathcal{I}_z^\delta F_\mu(z)| \geq \frac{|z|^{\mu+\delta}}{\Gamma(2+\delta)} \times \left(1 - \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1}\{(1-B)[(2\mu)^{k-1}\vartheta_2 - (2\mu)^{m-1}\lambda_2] + (A-B)(1-\gamma)(2\mu)^{k-1}\vartheta_2}\}|z|^\mu\right)$$

and

$$\mathcal{I}_z^\delta F_\mu(z) \leq \frac{|z|^{\mu+\delta}}{\Gamma(2+\delta)} \times \left(1 + \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1}\{(1-B)[(2\mu)^{k-1}\vartheta_2 - (2\mu)^{m-1}\lambda_2] + (A-B)(1-\gamma)(2\mu)^{k-1}\vartheta_2}\}|z|^\mu\right).$$

The results are sharp.

Proof. Let

$$\begin{aligned} \mathcal{F}(z) &= \Gamma(2+\delta)z^{-\delta}\mathcal{I}_z^\delta F_\mu(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(\mu n + 1)\Gamma(2-\delta)}{\Gamma(\mu n + 1 + \delta)} a_n z^{\mu n}, \end{aligned} \quad (3.6)$$

$$= z - \sum_{n=2}^{\infty} \Upsilon(n) a_n z^{\mu n} \quad (3.7)$$

where

$$\Upsilon(n) = \frac{\Gamma(\mu n + 1)\Gamma(2+\delta)}{\Gamma(\mu n + 1 + \delta)} \quad (n = 2, 3, \dots). \quad (3.8)$$

It is clear that $\Upsilon(n)$ is a decreasing function of n , we can write as

$$0 < \Upsilon(n) \leq \Upsilon(2) = \frac{2\mu(1)_{2\mu-1}}{(2+\delta)_{2\mu-1}}. \quad (3.9)$$

On the other hand, in view of Theorem 2, we obtain

$$\begin{aligned} &[(1-B)((2\mu)^k\vartheta_2 - (2\mu)^m\lambda_2) + (A-B)(1-\gamma)(2\mu)^k\vartheta_2] \sum_{n=2}^{\infty} a_n \\ &\leq \sum_{n=2}^{\infty} [(1-B)((\mu n)^k\vartheta_n - (\mu n)^m\lambda_n) + (A-B)(1-\gamma)(\mu n)^m\vartheta_2] a_n \\ &\leq (A-B)(1-\gamma). \end{aligned}$$

Then

$$\sum_{n=2}^{\infty} a_n \leq \frac{(A-B)(1-\gamma)}{[(1-B)((2\mu)^k\vartheta_2 - (2\mu)^m\lambda_2) + (A-B)(1-\gamma)(2\mu)^k\vartheta_2]} \quad (3.10)$$

Thus, by using (3.18) and (3.19), we see that

$$|\mathcal{F}(z)| \geq |z| - \Upsilon(2)|z|^{2\mu} \sum_{n=2}^{\infty} a_n \quad (3.11)$$

$$\geq |z| - \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1}[(1-B)((2\mu)^{k-1}\vartheta_2 - (2\mu)^{m-1}\lambda_2) + (A-B)(1-\gamma)(2\mu)^{k-1}\vartheta_2]}|z|^{2\mu} \quad (3.12)$$

Similarly,

$$|\mathcal{F}(z)| \leq |z| + \Upsilon(2)|z|^{2\mu} \sum_{n=2}^{\infty} a_n \quad (3.13)$$

$$\leq |z| + \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1}[(1-B)((2\mu)^{k-1}\vartheta_2 - (2\mu)^{m-1}\lambda_2) + (A-B)(1-\gamma)(2\mu)^{m-1}\vartheta_2]}|z|^{2\mu} \quad (3.14)$$

From, this prove Theorem 8. The equalities are attained for the function $F(z)$ given by

$$\begin{aligned} |\mathcal{I}_z^\delta F_\mu(z)| &= \frac{z^{\mu+\delta}}{\Gamma(2+\delta)} \\ &\times \left(1 - \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1}[(1-B)((2\mu)^{k-1}\vartheta_2 - (2\mu)^{m-1}\lambda_2) + (A-B)(1-\gamma)(2\mu)^{m-1}\vartheta_2]}|z|^{2\mu} \right) \end{aligned}$$

Then the result are sharp and the proof of Theorem 8 is completed. \square

Corollary 3. Under the hypothesis of Theorem 8, $\mathcal{I}_z^\delta F_\mu(z)$ is included in a disk with its center at the origin and radius \mathcal{R}_1 given by

$$\mathcal{R}_1 = \frac{1}{\Gamma(2+\delta)} \quad (3.15)$$

$$\times \left(1 - \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1}[(1-B)((2\mu)^{k-1}\vartheta_2 - (2\mu)^{m-1}\lambda_2) + (A-B)(1-\gamma)(2\mu)^{m-1}\vartheta_2]}|z|^{2\mu} \right) \quad (3.16)$$

Theorem 9. Let the function $F_\mu(z)$ given by (1.12) be in the class $\mathcal{E}_{k,m}(\Phi_\mu, \Psi_\mu, A, B, \gamma, \mu)$, then

$$\begin{aligned} |\mathcal{D}_z^\delta F_\mu(z)| &\geq \frac{|z|^{\mu-\delta}}{\Gamma(2-\delta)} \\ &\times \left(1 - \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1}[(1-B)((2\mu)^{k-1}\vartheta_2 - (2\mu)^{m-1}\lambda_2) + (A-B)(1-\gamma)(2\mu)^{m-1}\vartheta_2]}|z|^{2\mu} \right) \end{aligned}$$

and

$$\begin{aligned} |\mathcal{D}_z^\delta F_\mu(z)| &\leq \frac{|z|^{\mu-\delta}}{\Gamma(2-\delta)} \\ &\times \left(1 - \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1}[(1-B)((2\mu)^{k-1}\vartheta_2 - (2\mu)^{m-1}\lambda_2) + (A-B)(1-\gamma)(2\mu)^{m-1}\vartheta_2]}|z|^{2\mu} \right). \end{aligned}$$

Each f these results is sharp.

Proof. Let

$$\begin{aligned} \mathcal{G}(z) &= \Gamma(2-\delta)z^\delta \mathcal{I}_z^\delta F_\mu(z) \\ &= z - \sum_{n=2}^{\infty} \frac{\Gamma(\mu n)\Gamma(2-\delta)}{\Gamma(\mu n+1+\delta)}(\mu n)a_n z^{\mu n}, \\ &= z - \sum_{n=2}^{\infty} \Omega(n)(\mu n)a_n z^{\mu n} \end{aligned}$$

where

$$\Omega(n) = \frac{\Gamma(\mu n)\Gamma(2-\delta)}{\Gamma(\mu n+1-\delta)} \quad (n = 2, 3, \dots). \quad (3.17)$$

It is clear that $\Omega(n)$ is a decreasing function of n , we can write as

$$0 < \Omega(n) \leq \Omega(2) = \frac{(1)_{2\mu-1}}{(2-\delta)_{2\mu-1}}. \quad (3.18)$$

On the other hand, in view of Theorem 2, we obtain

$$\begin{aligned} & [(1-B) \left((2\mu)^k \vartheta_2 - (2\mu)^m \lambda_2 \right) + (A-B)(1-\gamma)(2\mu)^m \vartheta_2] \sum_{n=2}^{\infty} (\mu n) a_n \\ & \leq \sum_{n=2}^{\infty} [(1-B) \left((\mu n)^k \vartheta_n - (\mu n)^m \lambda_n \right) + (A-B)(1-\gamma)(\mu n)^m \vartheta_n] a_n \\ & \leq (A-B)(1-\gamma). \end{aligned}$$

Then

$$\sum_{n=2}^{\infty} (\mu n) a_n \leq \frac{(A-B)(1-\gamma)}{[(1-B) \left((2\mu)^{k-1} \vartheta_2 - (2\mu)^{m-1} \lambda_2 \right) + (A-B)(1-\gamma)(2\mu)^{m-1} \vartheta_2]} \quad (3.19)$$

Thus, by sing (3.18) and (3.19), we see that

$$|\mathcal{G}(z)| \geq |z| - \Omega(2)|z|^{2\mu} \sum_{n=2}^{\infty} (\mu n) a_n \quad (3.20)$$

$$\geq |z| - \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1} [(1-B) \left((2\mu)^{k-1} \vartheta_2 - (2\mu)^{m-1} \lambda_2 \right) + (A-B)(1-\gamma)(2\mu)^{m-1} \vartheta_2]} |z|^{2\mu} \quad (3.21)$$

Similarly,

$$|\mathcal{G}(z)| \leq |z| + \Omega(2)|z|^{2\mu} \sum_{n=2}^{\infty} (\mu n) a_n \quad (3.22)$$

$$\leq |z| + \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1} [(1-B) \left((2\mu)^{k-1} \vartheta_2 - (2\mu)^{m-1} \lambda_2 \right) + (A-B)(1-\gamma)(2\mu)^{m-1} \vartheta_2]} |z|^{2\mu} \quad (3.23)$$

From, this prove Theorem 9. The equalities are attained for the function $F_\mu(z)$ given by

$$\begin{aligned} |\mathcal{D}_z^\delta F_\mu(z)| &= \frac{z^{2\mu-\delta}}{\Gamma(2-\delta)} \\ &\times \left(1 - \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1} [(1-B) \left((2\mu)^{k-1} \vartheta_2 - (2\mu)^{m-1} \lambda_2 \right) + (A-B)(1-\gamma)(2\mu)^{m-1} \vartheta_2]} |z|^{2\mu} \right) \end{aligned}$$

Then the result are sharp and the proof of Theorem 8 is completed. \square

Corollary 4. Under the hypothesis of Theorem 9, $\mathcal{D}_z^\delta F_\mu(z)$ is included in a disk with its center at the origin and radius \mathcal{R}_2 given by

$$\mathcal{R}_1 = \frac{1}{\Gamma(2-\delta)} \quad (3.24)$$

$$\times \left(1 - \frac{(1)_{2\mu-1}(A-B)(1-\gamma)}{(2-\delta)_{2\mu-1}[(1-B)((2\mu)^{k-1}\vartheta_2 - (2\mu)^{m-1}\lambda_2) + (A-B)(1-\gamma)(2\mu)^{m-1}\vartheta_2]} \right) \quad (3.25)$$

4 Conclusion

In unit disk, we derived a new class of fractional power \mathcal{A}_μ and consider this class to define a generalized of many differential operator, also we employed this operator to define a new subclasses in open unit disk. Further, we studied some Characteristic properties and applications including certain fractional calculus operators.

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